Mathematical Methods for Computer Science II

Spring 2017

Series 4 – Hand in before Monday, 20.03.2017 - 13.00

1. The aim of this exercise is to illustrate the theorems and their proofs seen in class on partitions of integers, for instance using dot diagrams. You may use any reasonable value for n and k or take $n = 6$ and $k = 23$. Illustrate the following statements:
   a) $P_n(k) = p_n(k-n)$.
   b) $p_n(k) = p_{n-1}(k) + p_n(k-n)$.
   c) The number $P_n(k)$ of partitions of k into exactly n parts is equal to the number of partitions of k into parts whose maximum is n.
   d) The number $p_n(k)$ of partitions of k into at most n parts is equal to the number of partitions of k into parts which are all $\leq n$.

2. Suppose $a, b, c \in \mathbb{N}$. Prove that the number of partitions of $a-c$ into exactly $b-1$ parts, none exceeding c equals the number of partitions of $a-b$ into $c-1$ parts, none exceeding b.
   Hint: Illustrate a typical partition of the first type with a dot diagram and transform it into a typical partition of the second type using the transformations seen in class so as to obtain an one-to-one correspondence between both types.

3. Prove that the equation $y_1+2y_2+\ldots+ny_n = k$ has $p_n(k)$ nonnegative integer solutions $(y_1, \ldots, y_n) \in \mathbb{N}^n$.
   Hint: Start with $n = 3$ and illustrate the situation with a Ferrers diagram. Find a one-to-one correspondence between nonnegative integer solutions to the equation $x + 2y + 3z = k$ and $p_3(k)$.

4. * Prove the following equation for positive n, which helps computing the values of $p(n)$, the number of partitions of n into an arbitrary number of parts,
$$p(n) + \sum_{k\geq 1} (-1)^k \left(p\Big(n - \frac{k(3k-1)}{2}\Big) + p\Big(n - \frac{k(3k+1)}{2}\Big)\right) = 0,$$
   that is
$$p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + p(n-12) + p(n-15) - \ldots$$
   Hint: To prove this equation, first set $\Phi(z) := \prod_{i=1}^{\infty}(1-z^i)$, the inverse of the generating function of the sequence $(p(n))_{n\in\mathbb{N}}$. Expanding the product in $\Phi(z)$, derive that
$$\Phi(z) = \sum_{n\geq 1} (q_e(n) - q_o(n)) z^n,$$
   with $p_e(n)$, respectively $p_o(n)$, the number of partitions of n into an even, respectively odd, number of *distinct* parts. Finally use Euler's Pentagonal Number Theorem seen in class to conclude, noticing that
$$\Phi(z) = 1 + \sum_{k\geq 1} (-1)^k \left(z^{\frac{k(3k-1)}{2}} + z^{\frac{k(3k+1)}{2}} \right).$$

* Exercises with a $\star$ are intended for Discrete Mathematics II students only. However, MMI II students can gain additional bonus points by attempting them.