Université de Fribourg — Département de Mathématiques
Universität Freiburg — Mathematik Departement

Mathematical Methods for Computer Science I

Fall 2016

Series 3 – Hand in before Monday, 10.10.2016 - 13.00

1. **Definition:** A graph G is *regular* if the vertices of G have the same degree and is *regular of degree* r if this degree is r. Such graphs are also called r*-regular*.
   a) Construct a 4-regular and a 5-regular graph with 10 vertices.
   b) Prove that for integers r and n, there exists an r-regular graph with n vertices if and only if $0 \leq r \leq n-1$ and r and n are not both odd.

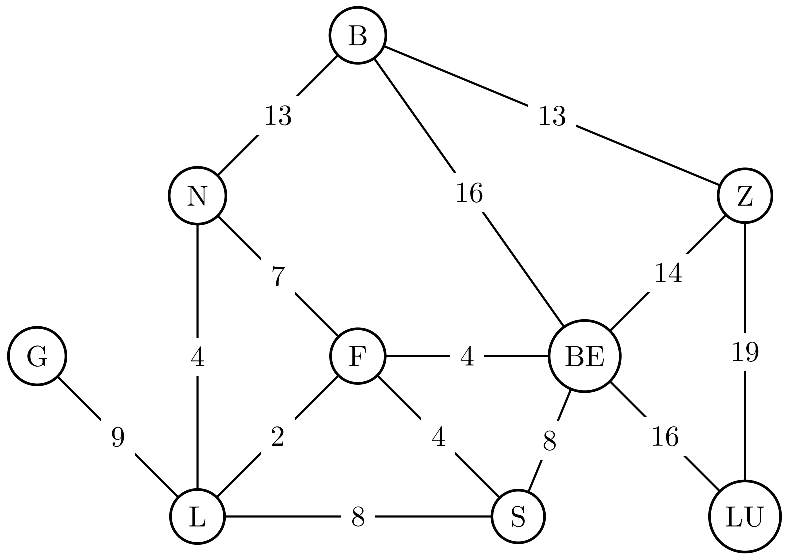
2. **Definitions:**
   - A *partial order* "$\leq$" on a set V is a relation which satisfies the three axioms:
     1. reflexivity: $v \leq v$ for all $v \in V$,
     2. anti-symmetry: $(v \leq w$ and $w \leq v) \Rightarrow v = w$ for all $v, w \in V$,
     3. transitivity: $(u \leq v$ and $v \leq w) \Rightarrow u \leq w$ for all $u, v, w \in V$.
   - A *maximal element* of the partial order $\leq$ is an element $m \in V$ with the property that if a vertex v satisfies $m \leq v$ then $v = m$.

   Let T be a nontrivial rooted tree with root v_0 (a root is a chosen particular vertex). For any vertices v and w of T, we denote vTw the unique path in T from v to w. For any vertices v and w of T, we denote $d(v, w)$ the length of the path vTw.

   a) Show that the relation $\leq$ defined on the set of vertices of T by
   $$v \leq w \Leftrightarrow v \in v_0Tw$$
   is a partial order.
   b) Show that $v \leq w$ implies $d(v_0, v) \leq d(v_0, w)$.
   c) Show that maximal elements for the partial order $\leq$ are exactly the leaves distinct from v_0.
   d) Show that the number of maximal elements is greater or equal to the number of neighbours of v_0.
   e) Conclude that the number of leaves of T is greater or equal to the degree of v_0.

3. (*Depth first search* in trees.) Let $G = (V, E)$ be a connected graph and $v_0 \in V$ be a vertex. Starting from v_0 move along the edges of G in such a way that the next vertex has not yet been visited. If there is no such vertex, backtrack until it becomes possible to do so and continue until all vertices have been visited.
   a) Show that the edges traversed in this way form a spanning tree T.
   b) If xy is an edge of G, show that x and y are comparable in the partial order on T given in Exercise 2. (Note: "x and y are comparable" means "$x \leq y$ or $y \leq x$".)

4. We consider the weighted connected graph (a *weighted graph* is a graph with a cost function) on the back of the sheet.
   a) Determine a minimum-cost spanning tree using Kruskal's algorithm. Explain intermediate steps.
   b) Determine a minimum-cost spanning tree using Prim's algorithm. Explain intermediate steps.
   c) Write a program in your preferred programming language that finds a minimum-cost spanning tree in a graph. Test it on the example of this exercise.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