Université de Fribourg
Universität Freiburg

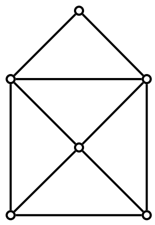
Département de Mathématiques
Mathematik Departement

Mathematical Methods for Computer Science I

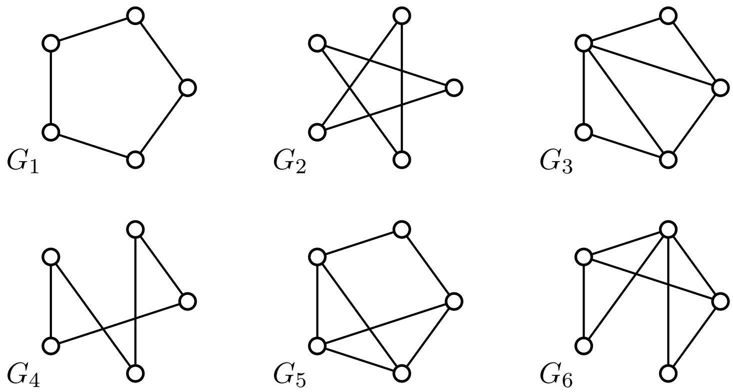
Fall 2016

Series 2 – Hand in before Monday, 3.10.2016 - 13:00

1. **Definition:** A **trail** in a graph is a walk that does not repeat any edges. If a trail in a graph G includes every edge of G, then that trail is said to be an **Eulerian trail**.
   a) Prove that a connected graph G contains an Eulerian trail if and only if there are at most two vertices of odd degree.
   b) Extend Fleury's algorithm so that it also finds an Eulerian trail in a graph if there is any, and apply it to the graph below.



2. Identify isomorphic graphs in the following list, briefly justify your answer.



3. i) Do there exist graphs having the following degree sequences? Prove your answers and give an example if the answer is "yes":
   a) (5, 4, 3, 3, 1, 1, 0)
   b) (4, 4, 4, 2, 2, 2)
   c) (6, 4, 3, 3, 2, 1)
   d) (5, 4, 3, 3, 3, 2)

   ii) Write a program in your preferred programming language that checks whether a given sequence of integers is the degree sequence of a graph. Test it on the above examples.

4. a) Let G be a graph whose edge set can be partitioned into circuits (that is: every edge belongs to exactly one of the circuits). Prove that every vertex of G has even degree.
   b) Prove that if a complete graph K_n decomposes into a union of edge disjoint Hamiltonian circuits, then n is odd. *Hint*: use a).
   c) Let $n \geq 2$ be an even integer. Show that K_n decomposes into a union of $\frac{n}{2}$ edge disjoint Hamiltonian paths (path through all the vertices). *Hint*: consider the vertices as evenly spaced around a circle and consider the path $n, 1, n-1, 2, n-2, \dots, \frac{n}{2}$; rotate the circle to obtain other paths.

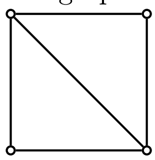
d) Let $n \geq 3$ be an odd integer. Show that K_n decomposes into a union of $\frac{n-1}{2}$ edge disjoint Hamiltonian circuits. *Hint*: use c).

e) Conclude: For $n \geq 3$ an integer, the complete graph K_n decomposes into a union of edge disjoint Hamiltonian circuits if and only if n is odd.

5. ⋆ **Definition:** The **adjacency matrix** $A(G)$ of a graph $G = (V, E)$ with vertex set $V = \{v_1, \ldots, v_n\}$ is defined as the $n \times n$ matrix $A(G) = (a_{ij})_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n}}$ such that

$$a_{ij} = \begin{cases} 1 \text{ if } (v_i, v_j) \in E \\ 0 \text{ if } (v_i, v_j) \notin E \end{cases}.$$

a) Establish the adjacency matrix of the graph below.



b) Justify that the matrix A is symmetric and has entries equal to 0 along its diagonal.

c) Prove that the (i, j) entry of $A(G)^k$, $k \geq 1$, is the number of different walks of length k between v_i and v_j.

⋆ Exercises with a ⋆ are intended for Discrete Mathematics I students only. However, MMI I students can gain additional bonus points by attempting them.