Mathematical Methods for Computer Science II

Spring 2017

Outline 7

3.7. Inclusion-Exclusion Principle.

Motivation:
Given n sets $A_1, \ldots, A_n$, how many elements are there in $A_1 \cup A_2 \cup \ldots \cup A_n = \bigcup_{i=1}^{n} A_i$?

Theorem. *For all sets A and B*

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

Theorem. *Inclusion-Exclusion Principle. For all sets $A_1, A_2, \ldots, A_n$,*

$$\begin{aligned}
\left|\bigcup_{i=1}^{n} A_i\right| &= \sum_{i=1}^{n} |A_i| - \sum_{\substack{i,j=1 \\ i<j}}^{n} |A_i \cap A_j| + \sum_{\substack{i,j,k=1 \\ i<j<k}}^{n} |A_i \cap A_j \cap A_k| - \ldots + (-1)^{n-1} |A_1 \cap A_2 \cap \ldots \cap A_n| \\
&= |A_1| + |A_2| + \ldots + |A_n| \\
&\quad -|A_1 \cap A_2| - |A_1 \cap A_3| - \ldots - |A_2 \cap A_3| - \ldots \\
&\quad +|A_1 \cap A_2 \cap A_3| + \ldots \\
&\quad \ldots \\
&\quad +(-1)^{n-1} |A_1 \cap A_2 \cap \ldots \cap A_n|.
\end{aligned}$$

Definition: A **derangement** of a list of elements is a perturbation of the list where none of the elements is mapped to itself.

Theorem. *The number of derangements of a list of n elements is*

$$D(n) = n! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + (-1)^n \frac{1}{n!}\right).$$