Université de Fribourg
Universität Freiburg

Département de Mathématiques
Mathematik Departement

Mathematical Methods for Computer Science II

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Outline 6

3.6. **Catalan numbers.**

Definition: The **Catalan number** c_n is the number of ways of putting brackets in a product of n factors $x_1, \dots, x_n$, each way corresponding to a computation of the product by successive multiplication of two numbers.

Definition: A **binary tree** is a planar tree whose nodes either have no child or two children. Such a tree with n leaves has $n-1$ inner nodes, $2n-1$ nodes in total and $2n-2$ edges.

Theorem. *The number of binary trees with n leaves is the Catalan number c_n.*

Theorem. *The number of triangulations of a convex polygon with $n+1$ vertices ($n \geq 2$) is the Catalan number c_n.*

Theorem. *The Catalan numbers can be computed by the recursive formula:*

$$c_n = \begin{cases} 0 & n = 0 \\ 1 & n = 1 \\ \sum_{k=1}^{n-1} c_k c_{n-k} & n \geq 2 \end{cases}$$

n	0	1	2	3	4	5	6	7	8	...
c_n	0	1	1	2	5	14	42	132	429	...

Theorem. *The generating function of the Catalan numbers is*

$$C(z) = \sum_{n=0}^{\infty} c_n z^n = \frac{1 - \sqrt{1-4z}}{2}.$$

Theorem. *The Catalan numbers are given by the explicit formula*

$$c_n = \frac{1}{n}\binom{2n-2}{n-1} \qquad n \geq 1.$$