Université de Fribourg
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Département de Mathématiques
Mathematik Departement

Mathematical Methods for Computer Science I

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Outline 6

Theorem (Brooks, 1941)**.** *Let G be a* connected *graph. If G is neither a complete graph nor a circuit of odd length, then*

$$\chi(G) \leq \Delta(G).$$

Definition: A graph G is called a **planar graph** if G can be drawn in the plane without any two of its edges crossing, or if G is isomorphic to such a graph.

Theorem (Four Color Theorem)**.** *Every* planar *graph is 4-colorable.*

Definition: For a graph $G = (V, E)$ and a nonnegative integer λ, the number of colorings with at most λ colors is denoted by $P(G, \lambda)$. For variable λ, $P(G, \lambda)$ is called the **chromatic polynomial**.

Theorem (Birkhoff, 1913)**.** *For a graph $G = (V, E)$,*

$$P(G, \lambda) = a_n \lambda^n + \ldots + a_1 \lambda + a_0$$

is a polynomial of degree $n = |V|$. The coefficients a_k are integers, with $a_n = 1$, $a_{n-1} = -|E|$, $a_0 = 0$, and alternating signs.

Proposition. *For the complete graph K_n,*

$$P(K_n, \lambda) = \lambda(\lambda - 1) \cdot \ldots \cdot (\lambda - (n - 1)).$$

Proposition. *Let G be a graph and let e be an edge of G. Call $G' = G \setminus \{e\}$ the graph where e is removed from G. Define the graph G'' to be the graph obtained from G by removing e, identifying the end vertices of e, and leaving only one copy of any resulting multiple edges. Then*

$$P(G, \lambda) = P(G', \lambda) - P(G'', \lambda)$$

Proposition. *For two disjoint graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, we have*

$$P(G_1 \cup G_2, \lambda) = P(G_1, \lambda) \cdot P(G_2, \lambda).$$

Proposition. *Let T be a* tree *with n vertices. Then*

$$P(T, \lambda) = \lambda(\lambda - 1)^{n-1}.$$