Mathematical Methods for Computer Science I

Fall 2016

Outline 5

Definition: For a bipartite graph $G = (U \cup W, E)$ and $S \subseteq U$, let $N(S) \subseteq W$ be the set of neighbours of the vertices in S (not necessarily matched neighbours). The condition

$$|N(S)| \geq |S| \text{ for all } S \subseteq U$$

is called **Hall's condition**.

Theorem (Hall, 1935). *Let $G = (U \cup W, E)$ be a bipartite graph. Then U can be matched (i.e. there is a perfect matching) to a subset of $W \iff$ Hall's condition is satisfied.*

Corollary. *If $G = (U \cup W, E)$ is bipartite with $|U| \leq |W|$ and Hall's condition is satisfied, then G contains a matching of size $|U|$ which is a maximum matching. If $|U| = |W|$, then such a matching is a perfect matching in G.*

Definition: Let $\mathcal{A} = \{A_1, \dots, A_n\}$ be a family of subsets of a set $\mathcal{X}$. A set $\{x_1, \dots, x_n\}$ with the property that $x_i \in A_i, x_i \neq x_j$ if $i \neq j$, is called a **set of distinct representatives (SDR)** of the family $\mathcal{A}$.

Theorem (Hall, 1935). *A family $\mathcal{A} = \{A_1, \dots, A_n\}$ of n sets has a SDR*

$$\iff \left| \bigcup_{i \in F} A_i \right| \geq |F| \quad \text{for every } F \subseteq \{1, \dots, n\}.$$

1.4. Coloring.

Definitions:

- A **vertex coloring** of a graph $G = (V, E)$ is a map $c : V \to S$ such that $c(v) \neq c(w)$ whenever v and w are adjacent.
- The elements of S are the **colors**.
- A graph G has a **k-coloring** if $|S| = k$ (k is not necessarily minimal). G is called **k-colorable**.
- The *smallest* k for which there exists a k-coloring is called the **chromatic number** of G and is denoted by $\chi(G)$. A graph with $\chi(G) = k$ is called **k-chromatic**.