Université de Fribourg
Universität Freiburg
Département de Mathématiques
Mathematik Departement

Mathematical Methods for Computer Science II

Spring 2017

Outline 4

3.4. Partitions of integers

Definition: A **partition of an integer** $k \in \mathbb{N}$ is a decomposition of k into a sum $k = x_1 + x_2 + \ldots + x_n$ $(n \in \mathbb{N})$ of positive integers $x_1, \ldots, x_n$ called **parts** of the partition, *not considering their order*. The parts are usually sorted in decreasing order $x_1 \geq x_2 \geq \ldots \geq x_n \geq 1$.

Notation: $p(k)$: number of partitions of k into an arbitrary number of parts
$P_n(k)$: number of partitions of k into exactly n parts
$p_n(k)$: number of partitions of k into at most n parts

Theorem. *If $n \leq k$, $P_n(k) = p_n(k-n)$.*

Theorem. *Recursive formula for $p_n(k)$*

$$p_n(k) = \begin{cases} 1 & \text{if } n = 0, k = 0, \\ 0 & \text{if } n = 0, k \neq 0 \\ p_{n-1}(k) + p_n(k-n) & \text{if } 0 < n \leq k \\ p_k(k) & \text{if } n > k \end{cases}$$

Theorem. *Recursive formula for $P_n(k)$*

$$P_n(k) = \begin{cases} 1 & \text{if } n = 0, k = 0, \\ 0 & \text{if } n = 0, k \neq 0 \\ P_{n-1}(k-1) + P_n(k-n) & \text{if } 0 < n \leq k \\ 0 & \text{if } n > k \end{cases}$$

Definition: The **transposed** or **dual** partition of a partition of k is obtained by transposing the rows and columns of the dot diagram of a partition.

Theorem. *1) The number $P_n(k)$ of partitions of k into exactly n parts is equal to the number of partitions of k into parts whose maximum is n.*

2) The number $p_n(k)$ of partitions of k into at most n parts is equal to the number of partitions of k into parts which are all $\leq n$.

Theorem. *The equation $y_1 + 2y_2 + \ldots + ny_n = k$ has $p_n(k)$ nonnegative integer solutions $(y_1, \ldots, y_n) \in \mathbb{N}^n$.*

Theorem. *For any $n \in \mathbb{N}$,*

1) the generating function of the sequence $(p_n(k))_{k \in \mathbb{N}}$ is

$$\sum_{k=0}^{\infty} p_n(k) z^k = \frac{1}{(1-z)(1-z^2) \cdot \ldots \cdot (1-z^n)} \qquad (|z| < 1).$$

2) The generating function of $(p(k))_{k \in \mathbb{N}}$ is given by the infinite product

$$\sum_{k=0}^{\infty} p(k) z^k = \prod_{i=1}^{\infty} \frac{1}{1-z^i} = \frac{1}{(1-z)(1-z^2) \cdot \ldots} \qquad (|z| < 1).$$

Theorem. *Euler's Pentagonal Number Theorem*
Let k be a nonnegative integer, and let $p_e(k)$, respectively $p_o(k)$, the number of partitions of k into an even, respectively odd, number of distinct parts. Then

$$p_e(k) - p_o(k) = \begin{cases} (-1)^n & \text{if } k = \frac{n(3n \pm 1)}{2} \\ 0 & \text{otherwise.} \end{cases}$$