Université de Fribourg
Universität Freiburg

Département de Mathématiques
Mathematik Departement

Mathematical Methods for Computer Science II

Spring 2017

Outline 2

Proposition. ***Symmetry property:*** $\binom{n}{k} = \binom{n}{n-k} \qquad k \leq n.$

Theorem. ***Recursive formula*** *for the binomial coefficients:*

$$\binom{n}{k} = \begin{cases} 1 & , k = 0 \\ 0 & , k \neq 0, n = 0 \\ \binom{n-1}{k} + \binom{n-1}{k-1} & , k \neq 0, n \neq 0 \end{cases}$$

Theorem. ***Binomial formula***

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

Corollary.

$$\sum_{k=0}^{n} \binom{n}{k} = \binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n} = 2^n$$

Unordered choices with repetition

Theorem. *The number of* unordered *lists of* k *not necessarily distinct objects (i.e. with repetition) from a set containing* n *objects is*

$$\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle = \frac{[n]^k}{k!} = \binom{n+k-1}{k} = \binom{n+k-1}{n-1} \qquad (n, k \in \mathbb{N})$$

Theorem. ***Recursive formula:***

$$\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle = \begin{cases} 1 & , k = 0 \\ 0 & , k \neq 0, n = 0 \\ \left\langle \begin{matrix} n-1 \\ k \end{matrix} \right\rangle + \left\langle \begin{matrix} n \\ k-1 \end{matrix} \right\rangle & , k \neq 0, n \neq 0 \end{cases}$$

Multinomial coefficients

Definition: A **division** of a set X is a *list* $(X_1, ..., X_r)$ of pairwise disjoint subsets (i.e. no repetition) of X whose union is the whole set X:

$$X = X_1 \cup X_2 \cup \ldots \cup X_r \qquad X_i \cap X_j = \emptyset, i \neq j.$$

Theorem. *Let* X *be a set of cardinality* n*, and* $k_1, ..., k_r$ *be nonnegative integers with* $k_1 + \ldots + k_r = n$.
The number of divisions $(X_1, \ldots, X_r)$ *of* X *s.t.* $|X_i| = k_i, i = 1, \ldots, r$*, is given by the* ***multinomial coefficient***

$$\frac{n!}{k_1! \cdot \ldots \cdot k_r!} = \binom{n}{k_1}\binom{n-k_1}{k_2}\binom{n-k_1-k_2}{k_3} \cdot \ldots \cdot \binom{n-k_1-k_2-\ldots-k_{r-1}}{k_r}.$$

Theorem. *Multinomial theorem*

$$(a_1 + \ldots + a_r)^n = \sum_{\substack{(k_1,\ldots,k_r)\in\mathbb{N}^r \\ k_1+\ldots+k_r=n}} \frac{n!}{k_1! \cdot \ldots \cdot k_r!} a_1^{k_1} \cdot \ldots \cdot a_r^{k_r}$$