Mathematical Methods for Computer Science II

Spring 2017

Outline 1

Part 3: Combinatorics

References:

- R. Stanley: *Enumerative Combinatorics, Volume I*, Cambridge University Press
- S. Lipschutz, M. Lipson, *Discrete Mathematics*, Schaum's outline
- J. Harris, J. Hirst, M. Mossinghoff: *Combinatorics and Graph Theory*, Springer

Remember: Unordered means the order does not matter, as in sets: $\{1,2\} = \{2,1\}$, as opposed to *ordered* lists: $(1,2) \neq (2,1)$.

3.1. **Ordered choices.**
Ordered choices with repetition

Theorem. *The number of lists $(a_1, \ldots, a_k)$ of k not necessarily distinct elements (i.e. with repetition) from a set of n elements is n^k $(n, k \in \mathbb{N})$.*

Definition: For a set X, the **power set** $\mathcal{P}(X)$ is the set of all subsets of X, including X itself and $\varnothing$.

Proposition. $|\mathcal{P}(X)| = 2^{|X|}$.

Theorem. *The number of lists $(a_1, \ldots, a_k)$ of k* distinct *elements from a set of n elements (i.e. without repetition) is given by the falling factorial*

$$[n]_k = n \cdot (n-1) \cdot \ldots \cdot (n-k+1) = \prod_{i=1}^{k} (n-i+1) \qquad (k, n \in \mathbb{N})$$

Special case: $[n]_0 = 1$.

Definition: A function $f : X \to Y$ is **injective** (or one-to-one) if $x_i \neq x_j \Rightarrow f(x_i) \neq f(x_j)$
A function $f : X \to Y$ is **surjective** (or onto) if for every element $y \in Y$ there is an $x \in X$ s.t. $y = f(x)$
A function $f : X \to Y$ is **bijective** if f is injective and surjective.

Theorem. *Let A and B be finite sets with $|A| = k, |B| = n$. Then the number of injective functions from A to B is $[n]_k$.*

Theorem. *The number of bijective functions $f : X \to X$, where $|X| = n$, is $[n]_n = n!$, $n! = n \cdot (n-1) \cdot \ldots \cdot 2 \cdot 1$; $0! = 1$.*

Ordered choices without repetition

Theorem. *The number of distributions of k* distinct *objects (no repetition) into n boxes, considering the order within each box, is given by the rising factorial*

$$[n]^k = n(n+1) \cdot \ldots \cdot (n+k-1) = \prod_{i=1}^{k} (n+i-1) \qquad (k, n \in \mathbb{N})$$

Special case: $[n]^0 = 1$.

3.2. **Unordered choices.**

Unordered choices without repetition

Theorem. *The number of* subsets *of cardinality k of a set of cardinality n is given by the* ***binomial coefficient***

$$\binom{n}{k} = \frac{[n]_k}{k!} = \frac{n!}{(n-k)! \cdot k!} \qquad (k, n \in \mathbb{N})$$

Special case: $\binom{n}{0} = 1$, $\binom{n}{k} = 0$ *if* $k > n$.