Université de Fribourg
Universität Freiburg

Département de Mathématiques
Mathematik Departement

Mathematical Methods for Computer Science II

Spring 2017

Outline 14

Definition: The **Herbrand universe** $D(F)$ of a closed formula F in Skolem form is the set of all variable-free terms that can be built from the components of F.
Inductive definition :

1) Every constant occurring in F is in $D(F)$. If F does not contain any constant, then $a \in D(F)$.
2) For every k-ary function symbol f in F, and for all terms $t_1, \dots, t_k$ in $D(F)$, $f(t_1, \dots, t_k) \in D(F)$.

Definition: Let F be a closed formula in Skolem form. A structure $\mathcal{A} = (\mathcal{U}_\mathcal{A}, \mathcal{I}_\mathcal{A})$ is called a **Herbrand structure** for F if:

1) $\mathcal{U}_\mathcal{A} = D(F)$
2) For every k-ary function symbol f in F and for all terms $t_1, \dots, t_k \in D(F)$, $f^\mathcal{A}(t_1, \dots, t_k) = f(t_1, \dots, t_k)$.

Theorem. *Let F be a closed formula in Skolem form. Then F is satisfiable $\Leftrightarrow$ F has a Herbrand model.*

Corollary (Löwenheim–Skolem)**.** *Every satisfiable formula in predicate logic has a model which is countable (but not necessarily finite).*

Definition: Let $F = \forall y_1 \forall y_2 \dots \forall y_n \, F^*$ be a closed formula in Skolem form. The **Herbrand expansion** $E(F)$ is defined as

$$E(F) = \{F^*[y_1/t_1] \dots [y_n/t_n] \,\big|\, t_1, \dots, t_n \in D(F)\}.$$

Theorem (Gödel–Herbrand–Skolem)**.** *For each closed formula F in Skolem form, F is satisfiable $\Leftrightarrow$ the set of formulas $E(F)$ is satisfiable as a set of formulas in* propositional logic*.*

Theorem (Herbrand)**.** *A closed formula in Skolem form is unsatisfiable $\Leftrightarrow$ there is a finite subset of $E(F)$ which is unsatisfiable in the sense of propositional logic.*

Theorem. *The unsatisfiability problem and the validity problem for formulas in predicate logic are semi-decidable.*