Université de Fribourg
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Département de Mathématiques
Mathematik Departement

Mathematical Methods for Computer Science II

Spring 2017

Outline 12

Definition: A **structure** is a pair $\mathcal{A} = (\mathcal{U}_\mathcal{A}, \mathcal{I}_\mathcal{A})$ where $\mathcal{U}_\mathcal{A}$ is a non-empty set, called the **ground set** or universe. $\mathcal{I}_\mathcal{A}$ is a mapping which maps
- k-ary predicate symbols to k-ary predicates on $\mathcal{U}_\mathcal{A}$
- k-ary function symbols to k-ary functions on $\mathcal{U}_\mathcal{A}$
- each variable x to an element of $\mathcal{U}_\mathcal{A}$.

Definition: Let F be a formula and $\mathcal{A} = (\mathcal{U}_\mathcal{A}, \mathcal{I}_\mathcal{A})$ be a structure. $\mathcal{A}$ is called **suitable** for F if $\mathcal{I}_\mathcal{A}$ is defined for all predicate symbols, function symbols and for all variables that occur free in F.

Definitions: For a given structure $\mathcal{A}$ for a formula F, we can have that the **truth-value** $\mathcal{A}(F)$ is true ($\mathcal{A}(F) = 1$) or false ($\mathcal{A}(F) = 0$).
·) If $\mathcal{A}(F) = 1$, then $\mathcal{A}$ is a **model** for F.
·) If every suitable structure for F is a model for F, then F is **valid**, denoted by $\models F$.
·) If there is at least one model for F, then F is **satisfiable**, otherwise **unsatisfiable**.

Definition: The **substitution** of u for a term x in a formula F is denoted by $F[x/u]$ and $\mathcal{A}_{[x/u]}(F)$ in the sense of $x^\mathcal{A} = u$ no matter whether $\mathcal{I}_\mathcal{A}$ is defined on x or not.

Theorem. *Let F and G be arbitrary formulas. Then*
1)
$$\neg\forall x\ F \equiv \exists x\ \neg F$$
$$\neg\exists x\ F \equiv \forall x\ \neg F$$
2) If x does not occur free in G, then
$$\begin{aligned}(\forall x\ F \wedge G) \equiv \forall x\ (F \wedge G) \quad &; \quad (G \wedge \forall x\ F) \equiv \forall x\ (F \wedge G)\\ (\forall x\ F \vee G) \equiv \forall x\ (F \vee G) \quad &; \quad (G \vee \forall x\ F) \equiv \forall x\ (F \vee G)\\ (\exists x\ F \wedge G) \equiv \exists x\ (F \wedge G) \quad &; \quad (G \wedge \exists x\ F) \equiv \exists x\ (F \wedge G)\\ (\exists x\ F \vee G) \equiv \exists x\ (F \vee G) \quad &; \quad (G \vee \exists x\ F) \equiv \exists x\ (F \vee G)\end{aligned}$$
3)
$$(\forall x\ F \wedge \forall x\ G) \equiv \forall x\ (F \wedge G)$$
$$(\exists x\ F \vee \exists x\ G) \equiv \exists x\ (F \vee G)$$
4)
$$\forall x \forall y\ F \equiv \forall y \forall x\ F$$
$$\exists x \exists y\ F \equiv \exists y \exists x\ F.$$